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Upstream MGD phenomenon in a jet-threaded cylindrical field

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An exact analytic solution is derived for the perturbation of a magnetic field while exposed to an immersed, axisymmetric, azimuthal, steady current source of arbitrary distribution in the presence of a slender, electrically conducting, independently permeated, compressible jet threading the axis of symmetry, subject to an equilibrium pressure balance. A further influence is the enclosure of the magnetic field by a coaxial cylindrical wall. The steady-state result invariably exhibits an infinite discrete superposition of axially decaying terms. In addition, there arise two admissible alternatives involving a fluid parameter λ (dependent on the flow speed, sound speed and both Alfvén speeds pertaining to the jet) together with a scale parameter $\chi(0)$ (equal to twice the ratio of the cross-sectional area of the jet to that of its externally enveloping field). Provided that λ exceeds $\chi(0)$, each element constituting the current distribution induces a stationary-wave contribution confined, as a consequence of an applied radiation condition, to the upstream domain, corresponding to an upstream-directed group velocity. However, if λ is exceeded by $\chi(0)$, this upstream wave is replaced by another decaying term, acting on both sides of every current constituent, like all other decaying terms.

1. Introduction

The existence of an upstream stationary wave along a plasma–magnetic field interface was first theoretically demonstrated by Savage (1967), who worked with a rectangular configuration comprising a thin layer of fully ionized, magnetically field-free, moving plasma sandwiched between two equal unbounded magnetic fields. Savage then went on to study the case of a plasma layer of finite thickness, and showed that a subsonic upstream wave occurs under certain flow conditions. During supersonic motion, downstream waves appear instead. If the field-free plasma is replaced by a field-traversed but incompressible conducting fluid (Savage 1970), then whenever a stationary wave appears on an interface, it does so only upstream. The generating sources are of specific types, viz. a pair of magnetic dipoles (Savage 1967) placed within the confining field, or an algebraically decaying pressure (Savage 1970) exerted along the interface. The solutions possess extra terms, normally in the form of Laplace integrals. Except for the fact that these are decaying along the stream direction, little else is revealed about them. The possibility that these integrals might furnish decaying waves has apparently been overlooked. Consequently, only the asymptotic forms of Savage's results are truly explicit.

The present paper somewhat complements a recent investigation (Chee-Seng 1972). The latter deals with the disturbance of an axisymmetric MGD configuration wherein a compressible conducting cylindrical jet of substantial thickness is both permeated internally and contained externally by stream-aligned magnetic fields, the external field being unbounded. The results obtained are all asymptotic. If the jet motion is elliptic (in the sense of McCune & Resler 1960), a further restriction on the flow parameters leads to an upstream stationary wave. Alternatively, with hyperbolic jet motion, one encounters an infinite train of stationary waves, all of which are, in the case of a supersonic super-Alfvénic jet, to be found downstream. In particular, the conclusions reached concur with, being in fact the cylindrical analogue of, Savage's observations (1967, 1970) if, respectively, the permeating magnetic field is reduced to zero for a field-free jet, or the sound speed is raised to infinity to attain incompressibility.

Chee-Seng's analysis does not cover the situation where the jet column is slender (thin compared with the extent of the surrounding field). Furthermore, it suffers from a defect similar to Savage's in not yielding ample details of decaying (i.e. error) terms. These shortcomings are all remedied in the present paper, which is devoted to the establishment of an exact solution consistent with a slender jet. Perturbations originate from an azimuthal source current having an arbitrary axisymmetric distribution. Largely responsible for enabling an exact evaluation is the closing of the external confining field by a coaxial cylindrical wall of finite radius. There are just two possibilities, relating to a flow parameter λ and a scale ratio $\chi(0)$. Provided that $\lambda > \chi(0)$, a stationary wave emerges upstream. But if $\lambda < \chi(0)$, no stationary wave exists. Other terms are involved, there being an infinity of these, all of which are non-wavelike but, in fact, strictly decaying exponentially with increasing axial distance, on both sides of any current cross-section. Across the plane containing such a cross-section, the displacement of a magnetic line of force is continuous. The upstream phenomenon ensues from a mathematical interpretation of a basic physical notion, precisely, a radiation condition incorporated in accordance with Lighthill (1960, 1965).

There is a separate class of plasma waves in bounded systems of the Kruskal–Schwarzschild type (Kruskal & Schwarzschild 1954), namely, small amplitude Alfvén and ion cyclotron waves (Stix 1957, 1958, 1962) formed during steady harmonic excitations of a column of cold, pressureless, perfectly conducting plasma with zero electron mass contained axially by a vacuum field.

2. Formulation

We consider an infinitely long, slim, cylindrical jet of non-gravitating, inviscid, perfectly electrically conducting, compressible fluid within which is trapped an axial magnetic field. Outside the jet, aligned with and confining it, is another magnetic field (of uniform strength B_0, say), occupying a layer of vacuous space. This is, in turn, bounded by a perfectly conducting, coaxial, cylindrical solid wall. Into this equilibrium configuration, small perturbations are initiated by a weak *azimuthal* current source, having an *arbitrary axisymmetric density distribution* $I(x,r,t)$, say, over a coaxial tubular conductor ($s_0(x) \leqslant r \leqslant s_1(x)$) stationed in the vacuum field. Throughout, x is the axial co-ordinate, r is the radial co-ordinate measured from the axis of symmetry, while t represents time. Variations generated within the vacuum field are, correspondingly, small and axisymmetric. In constructing the asymptotic solution, Chee-Seng (*op. cit.*) found it convenient to assume a finite length for the current distribution. For the present problem, to which an exact solution is sought, this restriction will not be imposed. However, should the current distribution extend to $x = \pm\infty$, then it is understood that its source function $I(x,r,t)$ must be appropriately well behaved at $x = \pm\infty$.

A significantly slender jet is governed by, effectively, one-dimensional equations involving x and t. Suppose that $\mathbf{H}$ is the total (i.e. perturbed) magnetic field trapped within the jet and $S = |\mathbf{S}|$ is its variable normal cross-sectional area. The Cowling–Walen frozen-field condition, which incorporates both the induction and divergence equations, is then expressible as

$$\frac{D}{Dt}\int_S \mathbf{H}.d\mathbf{S} = 0, \tag{2.1}$$

with D/Dt denoting differentiation following the motion. Suppose that, in the uniform equilibrium phase, H_0 is the axial magnetic field frozen into the jet, r_0 is its radius, U is its velocity in the positive-x direction, ρ_0 is its density and p_0 is its pressure. Let the corresponding small variations from these quantities be, respectively, h, η, u, ρ and p. In particular, $\eta = \eta(x,t)$ is the transverse distortion of the jet profile. From (2.1), then,

$$D[(H_0+h)(r_0+\eta)^2]/Dt = 0.$$

The continuity condition is

$$\frac{\partial}{\partial t}[(\rho_0+\rho)(r_0+\eta)^2]+\frac{\partial}{\partial x}[(\rho_0+\rho)(r_0+\eta)^2(U+u)] = 0$$

and the axial momentum equation is

$$(\rho_0+\rho)\frac{Du}{Dt}+\frac{\partial}{\partial x}\left(p+\frac{\mu\mathbf{H}^2}{8\pi}\right) = \frac{\mu}{4\pi}(H_0+h)\frac{\partial h}{\partial x},$$

μ being the magnetic permeability of the fluid. From now on, all nonlinear equations are linearized for sufficiently small perturbations. Thus, the last three equations simplify to

$$r_0\,Dh/Dt+2H_0\,D\eta/Dt = 0, \tag{2.2}$$

$$r_0\,D\rho/Dt+2\rho_0\,D\eta/Dt+\rho_0 r_0\,\partial u/\partial x = 0, \tag{2.3}$$

$$\rho_0\,Du/Dt+\partial p/\partial x = 0, \tag{2.4}$$

which we accompany with the equation

$$Dp/Dt = c^2 D\rho/Dt, \tag{2.5}$$

for the equilibrium sound speed c of the fluid. Amongst these four equations, only (2.2) includes magnetic effects, the other three equations being strictly non-magnetic.

To construct a unique steady-state solution associated with a steady source current, we accommodate a *radiation condition* in the manner of Lighthill (1960, 1965; see also Savage and Chee-Seng *op. cit.*) by dealing preliminarily with an exponentially growing current:

$$(4\pi/B_0)\, I(x,r,t) = e^{\epsilon t} J(x,r) \quad (\epsilon > 0). \tag{2.6}$$

Let us introduce the Fourier transform (indicated by an asterisk)

$$J^*(\alpha, r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} J(x,r)\, e^{-i\alpha x}\, dx, \tag{2.7}$$

whose inverse

$$J(x,r) = \int_{-\infty}^{\infty} J^*(\alpha, r)\, e^{i\alpha x}\, d\alpha. \tag{2.8}$$

According to Lighthill, any dependent variable must be allowed, in the unsteady development, to expand exponentially in step with the source. Thus, if ξ is the axisymmetric radial displacement of a magnetic line of force of the perturbed vacuum field exterior to the jet, then during the development stage,

$$\xi = e^{\epsilon t}\int_{-\infty}^{\infty} \xi^*(\alpha, r; \epsilon)\, e^{i\alpha x}\, d\alpha. \tag{2.9}$$

In this case, the transformed radiation equation governing $\xi^* = \xi^*(\alpha, r; \epsilon)$ is (cf. Chee-Seng *op. cit.*)

$$\frac{\partial^2 \xi^*}{\partial r^2} + \frac{1}{r}\frac{\partial \xi^*}{\partial r} - \left(\alpha^2 + \frac{1}{r^2}\right)\xi^* = J^*(\alpha, r), \tag{2.10}$$

an inhomogeneous Bessel equation. Whence, on accommodating the Wronskian relation

$$K_n(z)\, I_n'(z) - I_n(z)\, K_n'(z) = z^{-1} \tag{2.11}$$

for the modified Bessel functions $I_n(z)$ and $K_n(z)$ of order n, the method of variation of parameters yields

$$\xi^* = I_1(r\alpha)\left(A(\alpha,\epsilon) + \int^r K_1(\kappa\alpha)\, J^*(\alpha,\kappa)\,\kappa\, d\kappa\right) + K_1(r\alpha)\left(B(\alpha,\epsilon) - \int^r I_1(\kappa\alpha)\, J^*(\alpha,\kappa)\,\kappa\, d\kappa\right). \tag{2.12}$$

The values $A(\alpha, \epsilon)$ and $B(\alpha, \epsilon)$ may be determined from appropriate boundary conditions which we shall next proceed to formulate.

The internal magnetic pressure plus the fluid pressure within the jet must balance the external magnetic pressure exerted by the vacuum field across the interface. In particular, during the initial undisturbed equilibrium,

$$p_0 + \mu H_0^2/8\pi = B_0^2/8\pi. \tag{2.13}$$

Whereupon, in the perturbed state, the specified pressure condition is reducible to

$$p^* + \frac{\mu H_0}{4\pi} h^* = -\frac{B_0^2}{4\pi}\left(\frac{\partial \xi^*}{\partial r} + \frac{\xi^*}{r}\right) \quad \text{at} \quad r = r_0 \tag{2.14}$$

(cf. Chee-Seng) under Fourier transformation. Now, the frozen-field condition (2.1) implies that the jet column is actually an **H** tube of force. Consequently, continuity of the normal magnetic field component across the interface requires that

$$\eta^* = \xi^* \quad \text{at} \quad r = r_0. \tag{2.15}$$

Taking into account the fact that the Fourier transform, via (2.9), of $D\eta/Dt$ is $(\epsilon + i\alpha U)\eta^*$ in linearized motion, a combination of (2.2)–(2.5) leads to

$$\begin{aligned} r_0[(U-i\epsilon\alpha^{-1})^2-c^2][p^* + (\mu H_0/4\pi)h^*] \\ = 2\rho_0\{a^2[c^2-(U-i\epsilon\alpha^{-1})^2] - c^2(U-i\epsilon\alpha^{-1})^2\}\eta^*, \end{aligned} \tag{2.16}$$

in which $a = (\mu H_0^2/4\pi\rho_0)^{\frac{1}{2}}$ is an internal Alfvén speed for the jet. Let

$$a_0 = (B_0^2/4\pi\rho_0)^{\frac{1}{2}},$$

an interface Alfvén speed, and define a function λ of V: by

$$\lambda(V) = (2/a_0^2)[c^2V^2/(c^2-V^2) - a^2] \tag{2.17}$$

for any argument V. From now on, we assume $U \neq c$. Then, from (2.14)–(2.16), the pressure condition on ξ^* is

$$r_0 \partial\xi^*/\partial r + \xi^* + \xi^*\lambda(U - i\epsilon\alpha^{-1}) = 0 \quad \text{at} \quad r = r_0. \tag{2.18}$$

There is an independent condition, namely, that, along the perfectly conducting cylindrical wall of radius r_1, say, enclosing the vacuum field, this vacuum field has a zero normal component, so that

$$\xi^* = 0 \quad \text{at} \quad r = r_1. \tag{2.19}$$

To complete the determination of ξ^*, (2.18) and (2.19) must be applied to (2.12). In doing this, the recurrence relations

$$zI_1'(z) = zI_0(z) - I_1(z), \quad zK_1'(z) = -zK_0(z) - K_1(z) \tag{2.20}$$

are required. The algebra is highly involved, however, and we shall merely display the result obtained. Thus, within $r_0 < r < r_1$,

$$\begin{aligned} \xi^*(\alpha, r; \epsilon) = \int_r^{r_1} \frac{Q(r, r_0, \alpha; U - i\epsilon\alpha^{-1})}{Q(r_1, r_0, \alpha; U - i\epsilon\alpha^{-1})} \Psi(\kappa, r_1, \alpha) J^*(\alpha, \kappa)\kappa\, d\kappa \\ + \int_{r_0}^{r} \frac{Q(\kappa, r_0, \alpha; U - i\epsilon\alpha^{-1})}{Q(r_1, r_0, \alpha; U - i\epsilon\alpha^{-1})} \Psi(r, r_1, \alpha) J^*(\alpha, \kappa)\kappa\, d\kappa, \end{aligned} \tag{2.21}$$

where $$Q(r, r_0, \alpha; U - i\epsilon\alpha^{-1}) = \Phi(r, r_0, \alpha) - \lambda(U - i\epsilon\alpha^{-1})\Psi(r, r_0, \alpha), \tag{2.22}$$

with $$\Phi(r, r_0, \alpha) = r_0\alpha[K_0(r_0\alpha)I_1(r\alpha) + I_0(r_0\alpha)K_1(r\alpha)], \tag{2.23}$$

$$\Psi(r, r_0, \alpha) = K_1(r_0\alpha)I_1(r\alpha) - I_1(r_0\alpha)K_1(r\alpha). \tag{2.24}$$

3. A Green's function and its Fourier transform

In accordance with Lighthill's (1960, 1965) interpretation, the unique steady-state solution for ξ is obtained from (2.9) by, first, evaluating the integral representation and then letting $\epsilon \to 0+$. Precisely, substituting the expression (2.21) and using (2.7), we can express, during the steady state,

$$\xi = \int_{-\infty}^{\infty} dz \int_r^{r_1} G(x-z, r, \kappa) J(z, \kappa)\kappa\, d\kappa + \int_{-\infty}^{\infty} dz \int_{r_0}^{r} G(x-z, \kappa, r) J(z, \kappa)\kappa\, d\kappa, \tag{3.1}$$

where
$$G(x,r,\kappa) = \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} G^*(\alpha,r,\kappa;\epsilon)\, e^{i\alpha x}\, d\alpha \tag{3.2}$$

for $r_0 < r \leqslant \kappa < r_1$, and whose Fourier transform

$$G^*(\alpha,r,\kappa;\epsilon) = \frac{\Psi(\kappa,r_1,\alpha)}{2\pi} \frac{Q(r,r_0,\alpha;U-i\epsilon\alpha^{-1})}{Q(r_1,r_0,\alpha;U-i\epsilon\alpha^{-1})}. \tag{3.3}$$

Corresponding to a unit source current carried round a circular filament of radius κ and centre $(x,r) = (0,0)$,

$$J(x,r) = r^{-1}\,\delta(x)\,\delta(r-\kappa) \quad (r_0 < \kappa < r_1), \tag{3.4}$$

$\delta(x)$ being the Dirac delta function, we have

$$\xi = \begin{cases} G(x,r,\kappa) & \text{in} \quad r_0 < r \leqslant \kappa < r_1, & (3.5)\\ G(x,\kappa,r) & \text{in} \quad r_0 < \kappa \leqslant r < r_1, & (3.6)\end{cases}$$

which is indeed continuous at $r = \kappa$. Evidently, the quantity $G(x,r,\kappa)$ defined by (3.2) is a Green's function (or kernel) for the present problem, and in all subsequent analysis pertaining to it, we assume that the observation position $(x,r) \neq (0,\kappa)$, i.e. anywhere along the current loop (3.4).

To determine the Green's function $G(x,r,\kappa)$, we must first examine the complex (α) extension of its transform given by (3.3). Now, in view of (2.17) and (2.22), we note that

$$F(r,\alpha,U) \equiv (c^2-U^2)\,Q(r,r_0,\alpha;U)$$

is linear in U^2, so that $F(r,\alpha,U-i\epsilon\alpha^{-1})$ is a quadratic (polynomial) function of $i\epsilon\alpha^{-1}$. Whence, (3.3) can be expanded into a more accessible form:

$$G^*(\alpha,r,\kappa;\epsilon) = \frac{\Psi(\kappa,r_1,\alpha)}{2\pi} \frac{\alpha^2F(r,\alpha,U)-i\epsilon\alpha F_U(r,\alpha,U)-\frac{1}{2}\epsilon^2F_{UU}(r,\alpha,U)}{\alpha^2F(r_1,\alpha,U)-i\epsilon\alpha F_U(r_1,\alpha,U)-\frac{1}{2}\epsilon^2F_{UU}(r_1,\alpha,U)}, \tag{3.7}$$

the suffix U indicating a U derivative. Now it follows from theorem 1 (see appendix) that $\Psi(\kappa,r_1,\alpha)$ and $Q(r,r_0,\alpha;U)$ are analytic functions of α. Hence, (3.7) defines a meromorphic function of α whose only singularities in the complex plane are poles located at the zeros of the denominator. As $\epsilon \to 0+$, this denominator approaches $\alpha^2F(r_1,\alpha,U)$, which possesses two zeros coinciding at $\alpha = 0$ and additional α-zeros identical to those of $Q(r_1,r_0,\alpha;U)$ (cf. (A 5), appendix). Via theorems 2–4 all α-zeros of the latter function are fully definable by the following cases.

Case $\lambda(U) > \chi(0)$. Two real zeros at

$$\alpha = \pm|\alpha_0|, \tag{3.8}$$

plus an infinity of imaginary zeros at

$$\alpha = \pm i|\alpha_\nu| \quad (\nu = 1,\ldots,\infty), \tag{3.9}$$

governed by

$$|\kappa_1| < |\alpha_1| < |\kappa_2| < |\alpha_2| < \ldots < |\kappa_n| < |\alpha_n| < |\kappa_{n+1}| < \ldots, \tag{3.10}$$

$\pm i|\kappa_\nu|$ $(\nu = 1,\ldots,\infty)$ being the entire set of α-zeros (all imaginary, cf. Gray & Mathews 1922, chap. 7) of $\Psi(r_1,r_0,\alpha)$.

Case $\lambda(U) < \chi(0)$. Only imaginary zeros are admitted, viz. at

$$\alpha = \pm i|\alpha_0|, \quad \alpha = \pm i|\alpha_\nu| \quad (\nu = 1, \ldots, \infty), \tag{3.11}$$

with

$$|\alpha_0| < |\kappa_1| < |\alpha_1| < |\kappa_2| < \ldots < |\kappa_n| < |\alpha_n| < |\kappa_{n+1}| < \ldots . \tag{3.12}$$

Here,

$$\chi(0) = \frac{\Phi(r_1, r_0, 0)}{\Psi(r_1, r_0, 0)} = -1 - r_0 \frac{\partial \Psi}{\partial r_0}(r_1, r_0, 0) \Big/ \Psi(r_1, r_0, 0) \equiv 2r_0^2/(r_1^2 - r_0^2), \tag{3.13}$$

on application of (A 1), (A 4), (A 6) and (A 16) to (A 17). All such zeros of $Q(r_1, r_0, \alpha; U)$ are simple (i.e. distinct) and non-coincident with the origin $\alpha = 0$, about which they are clearly symmetric. [According to theorem 3, a double zero of Q emerges at $\alpha = 0$ whenever $\lambda(U) = \chi(0)$. It can be demonstrated that this double zero would give rise to a solution $G(x, r, \kappa) = O(x)$ as $|x| \to \infty$, a phenomenon which is incompatible with linearized theory. Henceforth, we avoid the situation $\lambda(U) = \chi(0)$, in which case, $F(r_1, 0, U) \neq 0$.]

The Fourier integral of (3.2) can be tackled for small positive ϵ before the limit at $\epsilon = 0+$ is attained. Obviously, when ϵ is sufficiently small, the denominator in expression (3.7) has all its α-zeros almost coincident with those of $\alpha^2 F(r_1, \alpha, U)$, viz. two zeros almost overlapping at $\alpha = 0$ plus simple zeros nearly coincident with the simple zeros (defined under (3.8)–(3.12)) of $Q(r_1, r_0, \alpha; U)$. The former are (approximately) determined by

$$\alpha = i\epsilon[F_U(r_1, 0, U) \pm (F_U^2(r_1, 0, U) - 2F(r_1, 0, U)\, F_{UU}(r_1, 0, U))^{\frac{1}{2}}]/2F(r_1, 0, U), \tag{3.14}$$

being slightly separated and $O(\epsilon)$. Regarding the latter zeros, the particular simple zero near any simple zero $\alpha = \alpha_\nu$ of $Q(r_1, r_0, \alpha; U)$ appears at

$$\alpha = \alpha_\nu + i\epsilon / V(\alpha_\nu),$$

where

$$V(\alpha_\nu) = \alpha_\nu F_\alpha(r_1, \alpha_\nu, U)/F_U(r_1, \alpha_\nu, U) \equiv \alpha_\nu Q_\alpha(r_1, r_0, \alpha_\nu; U)/Q_U(r_1, r_0, \alpha_\nu; U),$$

with $Q_\alpha \equiv \partial Q/\partial \alpha$. In the case of a real $\alpha_\nu = \pm|\alpha_0|$ (for $\lambda(U) > \chi(0)$, cf. (3.8)), $V(\alpha_\nu)$ is a ***group velocity*** associated with a wave dispersion relative to a stationary frame (see Chee-Seng 1972). Application of (2.22) and the appendix result (A 15) yields

$$V(\alpha_\nu) = -2\alpha_\nu^2 \int_{r_0}^{r_1} |\Psi(r_1, \kappa, \alpha_\nu)|^2 \kappa\, d\kappa / \lambda'(U) |\Psi(r_1, r_0, \alpha_\nu)|^2 = V(-\alpha_\nu), \tag{3.15}$$

on further accommodating the fact (A 9):

$$\Psi(r_1, r, \alpha) = \Psi(r_1, r, -\alpha). \tag{3.16}$$

From (2.17), $\lambda'(U) = 4Uc^4/a_0^2(c^2 - U^2)^2 > 0$ (since $U > 0$, corresponding to which $x > 0$ is the downstream region relative to the current loop of (3.4)). Thus, at the two real zeros $\pm|\alpha_0|$ for $\lambda(U) > \chi(0)$, the group velocity function

$$V(|\alpha_0|) = V(-|\alpha_0|) < 0. \tag{3.17}$$

However, at any symmetric pair of imaginary zeros $\pm i|\alpha_\nu|$, displayed in (3.9) or (3.11), we always have

$$V(i|\alpha_\nu|) = V(-i|\alpha_\nu|) > 0. \tag{3.18}$$

Summarizing, we conclude that, sufficiently near $\epsilon = 0+$, $G^*(\alpha, r, \kappa; \epsilon)$ always possesses an infinite set of pairs of almost symmetric, purely imaginary, simple poles at

$$\alpha = \pm i(|\alpha_\nu| \pm \epsilon/V(i|\alpha_\nu|)), \tag{3.19}$$

each such pair having undergone a slight, vertical upward displacement (cf. (3.18)), through the distance $\epsilon/V(i|\alpha_\nu|)$, from the respective positions $\pm i|\alpha_\nu|$. In accordance with (3.11), the suffix $\nu = 0, 1, \ldots, \infty$ for $\lambda(U) < \chi(0)$. However, when $\lambda(U) > \chi(0)$, then $\nu = 1, \ldots, \infty$; but to compensate for a deficiency of two simple imaginary poles, there arise two simple, slightly complex poles at

$$\alpha = \pm |\alpha_0| + i\epsilon/V(|\alpha_0|), \tag{3.20}$$

both situated, in view of (3.17), within the lower half-plane $\operatorname{Im} \alpha < 0$, close to the real positions $\pm |\alpha_0|$. To ensure that the pertinent upward displacement from the particular $-i|\alpha_\nu|$ position nearest the $\operatorname{Re}\alpha$ axis does not cross this axis, we require

$$\epsilon < |\alpha_0| \, V(i|\alpha_0|) \quad (\lambda(U) < \chi(0)), \quad \epsilon < |\alpha_1| \, V(i|\alpha_1|) \quad (\lambda(U) > \chi(0)).$$

In addition, there are, near $\alpha = 0$, two simple poles expressed by (3.14). This completes the enumeration of all singularities possessed by the meromorphic function $G^*(\alpha, r, \kappa; \epsilon)$.

4. An exact evaluation

We next proceed to establish the exact value of the Fourier integral of (3.2) via contour integration. Ultimately, this Fourier integral acquires only residue contributions from all singularities of $G^*(\alpha, r, \kappa; \epsilon)$. Now, it can be formally demonstrated that the residue contribution from each of the two simple poles expressed by (3.14) is $O(\epsilon)$. This is not surprising as (3.7) reveals that, as $\epsilon \to 0+$, $G^*(\alpha, r, \kappa; \epsilon)$ approaches analyticity at $\alpha = 0$. Consequently, both these poles, being virtually non-contributing in the limit, will be completely ignored from here on. For our analysis, let us concentrate only on the situation where

$$\lambda(U) > \chi(0),$$

and deduce corresponding effects at the end (also, at end of § 5) for the complementary case $\lambda(U) < \chi(0)$.

Before applying residue theory, we require certain contour deformations onto large circular arcs. Along these arcs, relevant asymptotic approximations hold. Employing the asymptotic formulae for the modified Bessel functions in (2.22)–(2.24), we have, as $|\alpha| \to \infty$ within $|\operatorname{Re}\alpha| > 0$,

$$\Psi(\kappa, r_1, \alpha) \sim -\exp[(r_1 - \kappa)\alpha \operatorname{sgn}(\operatorname{Re}\alpha)]/2(\kappa r_1)^{\frac{1}{2}} \alpha \operatorname{sgn}(\operatorname{Re}\alpha) \quad (r_1 > \kappa),$$

$$Q(r, r_0, \alpha; U - i\epsilon\alpha^{-1}) \sim \Phi(r, r_0, \alpha) \sim \tfrac{1}{2}(r_0/r)^{\frac{1}{2}} \exp[(r - r_0)\alpha \operatorname{sgn}(\operatorname{Re}\alpha)] \quad (r > r_0),$$

where $\operatorname{sgn}(\operatorname{Re}\alpha) = \pm 1$ for $\operatorname{Re}\alpha \gtrless 0$. Hence, substituting into (3.3), which is defined for $r_0 < r \leqslant \kappa < r_1$,

$$G^*(\alpha, r, \kappa; \epsilon) \sim -\exp[(r - \kappa)\alpha \operatorname{sgn}(\operatorname{Re}\alpha)]/4\pi(\kappa r)^{\frac{1}{2}} \alpha \operatorname{sgn}(\operatorname{Re}\alpha) \quad (|\operatorname{Re}\alpha| > 0). \tag{4.1}$$

However, along the imaginary axis $\operatorname{Re}\alpha = 0$,

$$\Psi(r_1, r_0, \alpha) \sim \sin[(r_1 - r_0)\operatorname{Im}\alpha]/(r_1 r_0)^{\frac{1}{2}}\operatorname{Im}\alpha,$$

$$Q(r_1, r_0, \alpha; U - i\epsilon\alpha^{-1}) \sim \Phi(r_1, r_0, \alpha) \sim (r_0/r_1)^{\frac{1}{2}}\cos[(r_1 - r_0)\operatorname{Im}\alpha].$$

Thus, in the far region, the α-zeros of $\Psi(r_1, r_0, \alpha)$ appear at

$$\pm i|\kappa_\nu| \sim \pm i\nu\pi/(r_1 - r_0), \tag{4.2}$$

and these are, consistent with the monotonicity implied by (3.10) or (3.12), interlaced with the positions

$$\pm i|\alpha_\nu| \sim \pm i(\nu + \tfrac{1}{2})\pi/(r_1 - r_0), \tag{4.3}$$

which obviously approximate those far poles covered by (3.19). Here, the positive integer ν takes appropriately large values. Suppose that Γ is a sufficiently large circular arc $\alpha = R_\nu e^{i\theta}$ ($|\theta| \neq \frac{1}{2}\pi$) which never (quite) intersects the $\operatorname{Im}\alpha$ axis, but is confined to either $\operatorname{Re}\alpha > 0$ or $\operatorname{Re}\alpha < 0$. Let the radius $R_\nu = \nu\pi/(r_1 - r_0)$. So as $\nu \to \infty$, $R_\nu \to \infty$. From (4.1),

$$4\pi(\kappa r)^{\frac{1}{2}}\left|\int_\Gamma G^*(\alpha, r, \kappa; \epsilon)\, e^{i\alpha x}\, d\alpha\right|$$
$$\sim \left|\int_\Gamma \exp\{-R_\nu[x\sin\theta + (\kappa - r)\,|\cos\theta|]\} \times \exp\{iR_\nu[x\cos\theta - (\kappa - r)\sin\theta\operatorname{sgn}(\cos\theta)]\}\, d\theta\right|, \tag{4.4}$$

which remains bounded under $|\theta| \neq \frac{1}{2}\pi$ as $\nu \to \infty$ if, and only if,

$$x\sin\theta + (\kappa - r)\,|\cos\theta| \geqslant 0 \quad \text{all along } \Gamma. \tag{4.5}$$

Let $(x, \kappa - r) = R(\cos\omega, \sin\omega)$, with $R \neq 0$. Since $\kappa \geqslant r$, therefore $0 \leqslant \omega \leqslant \pi$.

Suppose that $x \geqslant 0$, i.e. $0 \leqslant \omega \leqslant \frac{1}{2}\pi$. The permissible range of θ throughout which (4.5) holds is then

$$\text{either} \quad \tfrac{1}{2}\pi > \theta \geqslant -\omega, \quad \text{or} \quad \pi + \omega \geqslant \theta > \tfrac{1}{2}\pi.$$

Consider the two circular arcs totally described inside $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\alpha < 0$ such that they join $R_\nu e^{-i\omega}$ and $R_\nu e^{i(\pi+\omega)}$ to the two points $iR_\nu + 0\pm$ respectively (see figure 1). Along any portion of each of these two arcs, (4.5) is invariably satisfied by $\theta = \arg\alpha$. This is not so however along circular extensions of these arcs beyond their lower ends $R_\nu e^{-i\omega}$ and $R_\nu e^{i(\pi+\omega)}$. When $r = \kappa$ ($x > 0$), these two ends are at R_ν and $-R_\nu$; but if $x = 0$ ($r < \kappa$), they must be taken at $-iR_\nu + 0\pm$. Otherwise, the lower ends are simply restricted to the fourth and third quadrants. In particular, the integral involved in (4.4) is convergent when $\nu \to \infty$ if, with reference to figure 1, Γ is selected to be any one of the following four circular arcs: the quarter-circles Γ_1 and Γ_2; the two serrated arcs joining R_ν to $R_\nu e^{-i\omega}$, and $-R_\nu$ to $R_\nu e^{i(\pi+\omega)}$, in the case $r \neq \kappa$. Thus, with $\Gamma = \Gamma_1$, for instance,

$$4\pi(\kappa r)^{\frac{1}{2}}\left|\int_{\Gamma_1} G^*(\alpha, r, \kappa; \epsilon)\, e^{i\alpha x}\, d\alpha\right|$$
$$\lesssim \int_0^{\frac{1}{2}\pi+0-} \exp[-RR_\nu\sin(\theta+\omega)]\, d\theta = \left(\int_\omega^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi-\omega-0-}^{\frac{1}{2}\pi}\right)\exp[-RR_\nu\sin\theta]\, d\theta$$
$$< \left(\int_\omega^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi-\omega-0-}^{\frac{1}{2}\pi}\right)\exp[-2RR_\nu\theta/\pi]\, d\theta$$
$$= (\pi/2RR_\nu)\{\exp[-2\omega RR_\nu/\pi] + \exp[-(1 - 2\omega/\pi - 0-)RR_\nu] - 2\exp[-RR_\nu]\}.$$

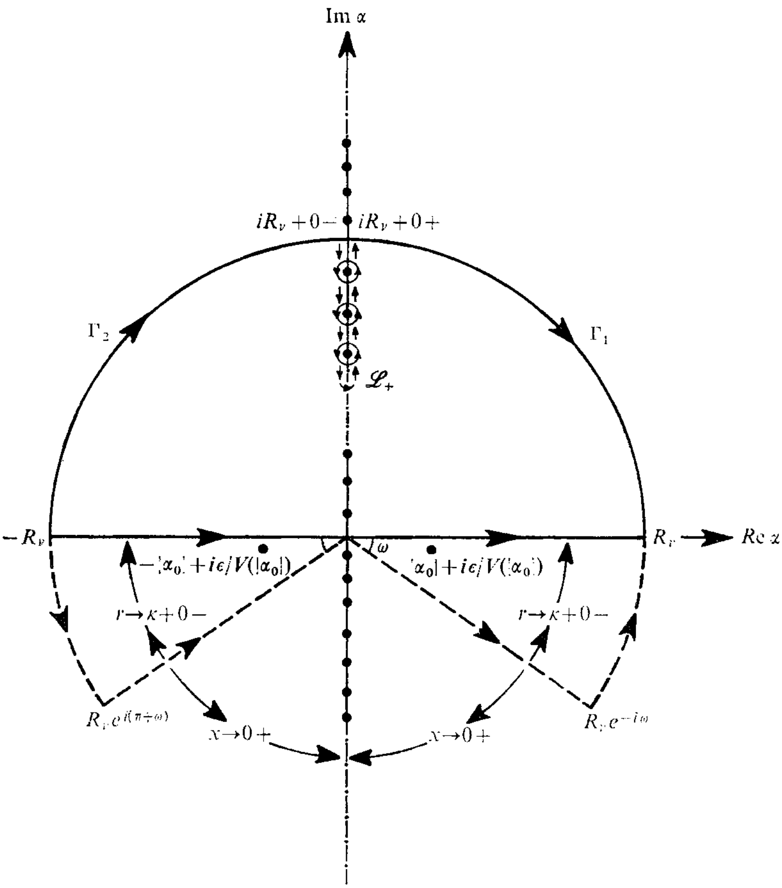

FIGURE 1. Alternative contour deformations for $x \geqslant 0$. As $x \to 0+$ $(r < \kappa)$, the two sectors bounded by the serrated deformed contour expand (arrow indications), with the edges $R_\nu e^{-i\omega}$ and $R_\nu e^{i(\pi+\omega)}$ approaching $-iR_\nu+0+$ and $-iR_\nu+0-$, respectively. However, if $r \to \kappa+0-$ $(x > 0)$, both these sectors gradually diminish, collapsing (at $r = \kappa$) onto the real interval $(-R_\nu, R_\nu)$. The imaginary poles are portrayed in their slightly asymmetric positions (3.19). Both near-real poles exist separately at $\pm|\alpha_0|+i\epsilon/V(|\alpha_0|)$ provided that $\lambda(U) > \chi(0)$ (but are converted into two imaginary poles $\pm i|\alpha_0|+i\epsilon/V(i|\alpha_0|)$ when $\lambda(U) < \chi(0)$). The two near-zero poles of (3.14) are not represented.

Whence, $\int_{\Gamma_1} \to 0$ as $\nu \to \infty$. Similarly, it can be proved that $\int_{\Gamma_2} \to 0$ as $\nu \to \infty$. To determine the integral $\int_{-\infty}^{\infty}$ of (3.2), we identify it as being $\lim_{\nu\to\infty}\int_{-R_\nu}^{R_\nu}$. For this purpose, the real path $(-R_\nu, R_\nu)$ may be deformed, when $x \geqslant 0$ and $\kappa \geqslant r$, into a complex contour comprising Γ_2, $\mathscr{L}_+$ and Γ_1 (arrowed as depicted in figure 1). The path Γ_2 is connected at $iR_\nu+0-$ to Γ_1 at $iR_\nu+0+$ via $\mathscr{L}_+$, a narrow vertical loop encircling and indented with small semicircles (on both sides) about all those imaginary poles of (3.19) which fall below the level of iR_ν, but are high enough in $\operatorname{Im}\alpha > 0$ to conform to the first approximation given in (4.3). As ν increases over positive integral values, the number of imaginary poles thus engulfed grows. During this process, the two ends of the loop $\mathscr{L}_+$ always hurdle across the level of any such incoming pole since $iR_\nu = i\nu\pi/(r_1-r_0) \mp i|\alpha_\nu|$, but $\sim i|\kappa_\nu|$ (cf. (4.2) and (4.3)). So all indentations on either side of and about these poles must indeed be full semicircles. Consequently, invoking residue theory,

$$\int_{-R_\nu}^{R_\nu} = \int_{\Gamma_2} + \int_{\mathscr{L}_+} + \int_{\Gamma_1} + 2\pi i \text{ (relevant residues)}$$
$$= \int_{\Gamma_2} + \int_{\Gamma_1} + 2\pi i \sum_{|\alpha_\nu|<R_\nu} \underset{i|\alpha_\nu|+i\epsilon/V(i|\alpha_\nu|)}{\text{residue}} \{G^*(\alpha, r, \kappa; \epsilon)\, e^{i\alpha x}\}$$

because the integrations over all straight segments of $\mathscr{L}_+$ cancel, so that $\int_{\mathscr{L}_+}$ accumulates only residue contributions from those far simple poles circumscribed by $\mathscr{L}_+$ below iR_ν. Therefore, letting $\nu \to \infty$, we obtain, in terms of the semi-infinite set of simple imaginary poles within $\operatorname{Im}\alpha > 0$,

$$\int_{-\infty}^{\infty} = 2\pi i \sum_{\nu=1}^{\infty} \underset{i|\alpha_\nu|+i\epsilon/V(i|\alpha_\nu|)}{\text{residue}} \{G^*(\alpha, r, \kappa; \epsilon)\, e^{i\alpha x}\} \quad (x \geqslant 0), \tag{4.6}$$

whenever $\lambda(U) > \chi(0)$. Alternatively, provided that $r \neq \kappa$, the path $(-R_\nu, R_\nu)$ may be deformed into the serrated contour (arrowed) composed of the circular arc $(-R_\nu, R_\nu e^{i(\pi+\omega)})$, the two straight paths $(R_\nu e^{i(\pi+\omega)}, 0)$ and $(0, R_\nu e^{-i\omega})$, plus another circular arc $(R_\nu e^{-i\omega}, R_\nu)$. This choice, however, leads to an apparently different form for $\int_{-\infty}^{\infty}$, consisting of two residue contributions from the two near-real poles (of (3.20)) within $\operatorname{Im}\alpha < 0$, together with two awkward complex integrals over the semi-infinite straight contours $(\infty \times e^{i(\pi+\omega)}, 0)$ and $(0, \infty \times e^{-i\omega})$. As it stands, then, this particular form is not adequately explicit, but may, of course, be equated with the result (4.6).

We next turn our attention to the case when $x \leqslant 0$, i.e. $\frac{1}{2}\pi \leqslant \omega \leqslant \pi$. The condition (4.5) is now, effectively,

$$\text{either} \quad \pi - \omega \geqslant \theta > -\tfrac{1}{2}\pi, \quad \text{or} \quad \tfrac{3}{2}\pi > \theta \geqslant \omega;$$

i.e. the contour Γ must be any portion of either the circular arc joining $R_\nu e^{i(\pi-\omega)}$ to $-iR_\nu + 0+$, or the circular arc joining $-iR_\nu + 0-$ to $R_\nu e^{i\omega}$ (see figure 2). If $\kappa > r$, we are again faced with a selection of two possible contour deformations of the integral path for $\int_{-R_\nu}^{R_\nu}$. Unless $\kappa = r$, we can, for example, deform $(-R_\nu, R_\nu)$ onto the serrated path drawn in figure 2. But this will merely lead to just two integrals performed along the straight paths $(\infty \times e^{i\omega}, 0)$ and $(0, \infty \times e^{i(\pi-\omega)})$. To achieve the desired effect, we deform, for $x \leqslant 0$ and $\kappa \geqslant r$, $(-R_\nu, R_\nu)$ into the lower half-plane $\operatorname{Im}\alpha < 0$ until we arrive at the contour comprising the two quarter-circle arcs Γ_3 and Γ_4, completed by the narrow vertical indented loop $\mathscr{L}_-$ circumscribing relevant imaginary poles. Together with $(-R_\nu, R_\nu)$, this deformed contour encloses the remaining imaginary poles inside $-R_\nu < \operatorname{Im}\alpha < 0$ as well as the two near-real poles $\pm|\alpha_0| + i\epsilon/V(|\alpha_0|)$. As for Γ_1, it can also be shown that $\int_{\Gamma_3}$ and $\int_{\Gamma_4}$ approach zero as $\nu \to \infty$. Moreover, $\int_{\mathscr{L}_-}$ can be related to the appropriate

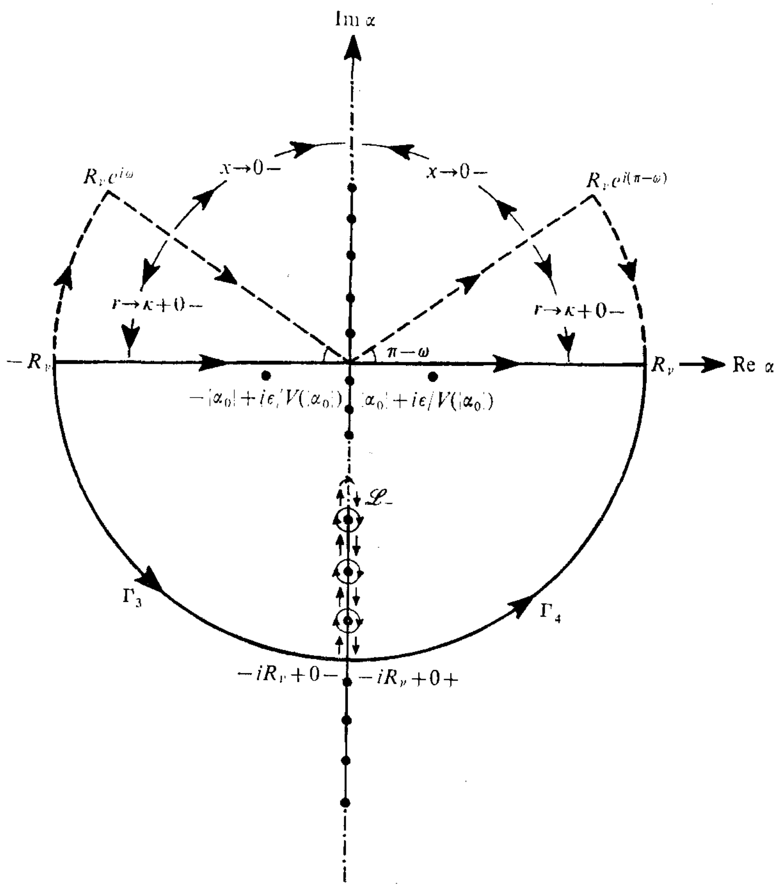

FIGURE 2. Alternative contour deformations for $x \leqslant 0$. The situation as $x \to 0-$ ($r < \kappa$), or $r \to \kappa + 0-$ ($x < 0$) is clearly traced by curved arrows. Again, the simple, near-real poles at $\pm|\alpha_0| + i\epsilon/V(|\alpha_0|)$ are present only if $\lambda(U) > \chi(0)$.

residues at those simple poles encircled by $\mathscr{L}_-$. Whereupon, we obtain for the integral in (3.2)

$$\int_{-\infty}^{\infty} = -2\pi i \Big(\underset{|\alpha_0|+i\epsilon/V(|\alpha_0|)}{\text{residue}} + \underset{-|\alpha_0|+i\epsilon/V(|\alpha_0|)}{\text{residue}} \Big) \{G^*(\alpha, r, \kappa; \epsilon)\, e^{i\alpha x}\}$$
$$-2\pi i \sum_{\nu=1}^{\infty} \underset{-i|\alpha_\nu|+i\epsilon/V(i|\alpha_\nu|)}{\text{residue}} \{G^*(\alpha, r, \kappa; \epsilon)\, e^{i\alpha x}\} \quad (x \leqslant 0), \tag{4.7}$$

wherein $\sum_{\nu=1}^{\infty}$ ranges over the semi-infinite set of simple poles along the negative $\operatorname{Im}\alpha$ axis, viz. $-i(|\alpha_\nu| - \epsilon/V(i|\alpha_\nu|))$ ($\nu = 1, \ldots, \infty$) with $|\alpha_1|\ V(i|\alpha_1|) > \epsilon$. Again, $\lambda(U) > \chi(0)$.

If $\lambda(U) < \chi(0)$, the two near-real poles $\pm|\alpha_0| + i\epsilon/V(i|\alpha_0|)$ are absent. In their place, we get two extra imaginary simple poles, viz. one at $i(|\alpha_0| + \epsilon/V(i|\alpha_0|))$ within $\operatorname{Im}\alpha > 0$, and one inside $\operatorname{Im}\alpha < 0$ at $-i(|\alpha_0| - \epsilon/V(i|\alpha_0|))$; $|\alpha_0|\ V(i|\alpha_0|) > \epsilon$. There is no staggering modification to the above analysis. Pertinent deviations from the expressions (4.6) and (4.7) are not difficult to envisage (see last paragraph of § 5).

5. Exact results and interpretations

The Green's function, as defined by (3.2), is now obtained for the case $\lambda(U) > \chi(0)$, by essentially putting $\epsilon = 0$ in (4.6) and (4.7) and formulating the prescribed residues. As each of these is associated with merely a simple zero $\alpha = \alpha_\nu$ of $Q(r_1, r_0, \alpha; U)$, then $\underset{\alpha_\nu}{\text{residue}}\{G^*(\alpha, r, \kappa; 0)\, e^{i\alpha x}\} = e^{i\alpha_\nu x}\, \underset{\alpha_\nu}{\text{residue}}[\;]$, where, via (3.3), (3.16) and the fact that $Q(r, r_0, \alpha; U) = Q(r, r_0, -\alpha; U)$ (cf. (A 9)), $\underset{\alpha_\nu}{\text{residue}}[\;]$ is an odd function of α_ν, determinable from (3.3). Whence, the combination of (4.6) and (4.7) leads to

$$G(x, r, \kappa) = \frac{2Q(r, r_0, |\alpha_0|; U)}{Q_\alpha(r_1, r_0, |\alpha_0|; U)}\Psi(\kappa, r_1, |\alpha_0|)\sin(|\alpha_0|\, x)\, H(-x) - \sum_{\nu=1}^{\infty} \frac{Q(r, r_0, i|\alpha_\nu|; U)\,\Psi(\kappa, r_1, i|\alpha_\nu|)}{[\partial Q(r_1, r_0, i\beta; U)/\partial\beta]_{\beta=|\alpha_\nu|}}\, e^{-|\alpha_\nu||x|}, \tag{5.1}$$

$H(-x)$ being the Heaviside unit function, equal to 1 ($x \leqslant 0$) or 0 ($x > 0$), and the suffix α denoting an α derivative. Our steady-state Green's function is thus composed of *an infinite discrete superposition of terms which are strongly decaying like* $e^{-|\alpha_\nu||x|}$ ($\nu = 1, \ldots, \infty$) as $x \to \pm\infty$, together with *a stationary sine wave which is encountered only upstream* (i.e. in $x < 0$) *of the source filament* (3.4). *This stationary wave* is of wavelength $2\pi/|\alpha_0|$ and has an amplitude dependent on r, and the reason why it *exists strictly upstream* is clearly *because its sustaining energy, which is propagated with the group velocity* $V(|\alpha_0|)$, *is*, in view of (3.17), *permanently convected upstream*. This upstream phenomenon is, of course, *a consequence of the applied radiation condition*. The sine wave gradually disappears as $x \to 0-$, so that the expression (5.1) is indeed continuous across $x = 0$.

The transverse distortion ξ of a magnetic line of force of the vacuum field is uniquely determined in the steady state by applying the solution (5.1) to (3.1). Thus, for $\lambda(U) > \chi(0)$,

$$\begin{aligned}\xi = {} & \frac{2Q(r, r_0, |\alpha_0|; U)}{Q_\alpha(r_1, r_0, |\alpha_0|; U)}\int_r^{r_1}\Psi(\kappa, r_1, |\alpha_0|)\,\kappa\, d\kappa \int_x^\infty J(z, \kappa)\sin[|\alpha_0|\,(x-z)]\, dz \\ & + \frac{2\Psi(r, r_1, |\alpha_0|)}{Q_\alpha(r_1, r_0, |\alpha_0|; U)}\int_{r_0}^{r} Q(\kappa, r_0, |\alpha_0|; U)\,\kappa\, d\kappa \int_x^\infty J(z, \kappa)\sin[|\alpha_0|\,(x-z)]\, dz \\ & - \sum_{\nu=1}^{\infty}\frac{Q(r, r_0, i|\alpha_\nu|; U)}{[\partial Q(r_1, r_0, i\beta; U)/\partial\beta]_{\beta=|\alpha_\nu|}}\int_r^{r_1}\Psi(\kappa, r_1, i|\alpha_\nu|)\,\kappa\, d\kappa\int_{-\infty}^{\infty} J(z, \kappa)\, e^{-|\alpha_\nu||x-z|}\, dz \\ & - \sum_{\nu=1}^{\infty}\frac{\Psi(r, r_1, i|\alpha_\nu|)}{[\partial Q(r_1, r_0, i\beta; U)/\partial\beta]_{\beta=|\alpha_\nu|}}\int_{r_0}^{r} Q(\kappa, r_0, i|\alpha_\nu|; U)\,\kappa\, d\kappa\int_{-\infty}^{\infty} J(z, \kappa)\, e^{-|\alpha_\nu||x-z|}\, dz.\end{aligned} \tag{5.2}$$

Again, there exists an infinite superposition of strong, axially dissipative elements, accompanied by *a stationary sine–cosine wave* of wavelength $2\pi/|\alpha_0|$. At any point (x, r), this stationary wave is governed by *an integral convolution with only the downstream* ($z > x$) *part of the source function* $J(z, \kappa)$, i.e. *every cross-section of the source current distribution induces an upstream stationary-wave contribution.*

Regarding both (5.1) and (5.2), the wavenumber $|\alpha_0|$ is the (only) real and positive α-zero (necessarily simple) of $Q(r_1, r_0, \alpha; U)$. The quantities

$$\Psi(r, r_1, |\alpha_0|) \quad \text{and} \quad Q(r, r_0, |\alpha_0|; U)$$

are real and obtainable from (2.22)–(2.24). The quantity $\Psi(r, r_1, i|\alpha_\nu|)$ is real by virtue of (A 11); see appendix. In particular (cf. (A 19)),

$$\Psi(r, r_1, i|\alpha_\nu|) = \tfrac{1}{2}\pi[J_1(r_1|\alpha_\nu|)\,Y_1(r|\alpha_\nu|) - Y_1(r_1|\alpha_\nu|)\,J_1(r|\alpha_\nu|)], \tag{5.3}$$

J_ν and Y_ν representing the fundamental Bessel functions. Also, $\sum_{\nu=1}^{\infty}$ is performed over all real, positive β-zeros at $\beta = |\alpha_\nu|$ $(\nu = 1, \ldots, \infty)$ of $Q(r_1, r_0, i\beta; U)$, which is real for real β and can be shown to be given by

$$\tfrac{1}{2}\pi r_0\beta[Y_0(r_0\beta)\,J_1(r_1\beta) - J_0(r_0\beta)\,Y_1(r_1\beta)] \\ - \tfrac{1}{2}\pi\lambda(U)\,[J_1(r_0\beta)\,Y_1(r_1\beta) - Y_1(r_0\beta)\,J_1(r_1\beta)]. \tag{5.4}$$

Its infinite set of positive zeros at $\beta = |\alpha_1|, |\alpha_2|, \ldots$, are all simple and satisfy (3.10). It is now obvious that the form (5.1) is real. So is the form (5.2), provided that the source function $J(x, r)$ is real.

Each contribution to ξ is a double integral transform of the source function. In particular, the stationary-wave member of ξ is a combination of finite I and K transforms of a Fourier sine transform of $J(x+z, \kappa)$, viz.

$$\int_0^\infty J(x+z, \kappa) \sin(|\alpha_0|\, z)\, dz.$$

Likewise, the exponentially decaying $|\alpha_\nu|$ element is expressible as a combination of finite Hankel and Y transforms of the Laplace transform

$$\int_0^\infty [J(x-z, \kappa) + J(x+z, \kappa)]\, e^{-|\alpha_\nu| z}\, dz.$$

This is a Laplace transform (resulting from a convolution aspect) of an arbitrary forcing effect which can admit wave functions of its own accord, but does not induce additional stationary waves (i.e. of the magneto-acoustic type encountered all along) associated strictly with the given MGD system. The situation is quite different from that of Savage (1970, also 1967), wherein his 'local disturbances' are formulated as Laplace transforms of specific functions which are, effectively, the combination of a Fourier transform of the particular forcing effect used, as well as other factors. The latter may, or may not, generate magneto-acoustic waves that decay in the flow direction. The present study of the cylindrical case suggests the second possibility.

The solutions (5.1) and (5.2) are exact. Suppose that the source function $J(x, r) = X(x)\, Y(r)$ and is distributed over a finite axial distance of $2l$: $X(x) \equiv 0$ in $|x| > l$. Then the solution (5.2) behaves in the *far field* as

$$\xi = (\text{known function of } r) \times \int_{-l}^{l} X(z) \sin[|\alpha_0|\,(x-z)]\, dz + O(e^{|\alpha_1| x})$$

for $x \ll -l$, while $\xi = O(e^{-|\alpha_1| x})$ for $x \gg l$.

So far, all results assembled are valid only when $\lambda(U) > \chi(0)$, where $\lambda(U)$ and $\chi(0)$ are given respectively by (2.17) and (3.13). Note that, with reference to the initial undisturbed configuration, $\chi(0) = 2$ (cross-sectional area of jet)/(cross-sectional area of vacuum field). As indicated in § 3, there is no acceptable solution for $\lambda(U) = \chi(0)$. On the other hand, suppose that $\lambda(U) < \chi(0)$. In this case, $Q(r_1, r_0, \alpha; U)$ has no real zeros, so that no (real) stationary wave is observed anywhere. However, $Q(r_1, r_0, i\beta; U)$ has one additional positive zero at $\beta = |\alpha_0|$. This zero is simple and satisfies (3.12). It gives rise to a further and stronger exponential decay $O(e^{-|\alpha_0||x|})$. Hence, briefly, the corresponding results for $\lambda(U) < \chi(0)$ are derived by eliminating the sine wave in (5.1), as well as the sine–cosine wave in (5.2), and substituting $\sum_{\nu=0}^{\infty}$, which starts from the index $\nu = 0$, for $\sum_{\nu=1}^{\infty}$. The asymptotic development of ξ, associated with the current source of restricted length $2l$, is now dominated, within $|x| \gg l$, by the factor $e^{-|\alpha_0||x|}$.

Appendix

Let

$$P(r,\alpha) = \phi(r,\alpha) - \lambda\psi(r,\alpha), \tag{A 1}$$

where

$$\phi(r,\alpha) = r\alpha[K_0(r\alpha)\, I_1(r_1\alpha) + I_0(r\alpha)\, K_1(r_1\alpha)], \tag{A 2}$$

$$\psi(r,\alpha) = K_1(r\alpha)\, I_1(r_1\alpha) - I_1(r\alpha)\, K_1(r_1\alpha), \tag{A 3}$$

with $r \leqslant r_1 < \infty$ and $I_n(\alpha)$ and $K_n(\alpha)$ being the modified Bessel functions. In relation to (2.22)–(2.24), we see that

$$\phi(r,\alpha) \equiv \Phi(r_1, r, \alpha), \quad \psi(r,\alpha) \equiv \Psi(r_1, r, \alpha), \tag{A 4}$$

while

$$P(r,\alpha) \equiv Q(r_1, r, \alpha; U) \tag{A 5}$$

if, in (A 1), the parameter $\lambda \equiv \lambda(U)$ defined by (2.17). The objective of this appendix is to determine appropriate properties of $\psi(r,\alpha)$ and $P(r,\alpha)$ required in dealing with a Green's function (§ 3). First, we note that, via the recurrence relations of (2.20), (A 1)–(A 3) produce

$$(\lambda+1)\,\psi(r,\alpha) + r\,\partial\psi(r,\alpha)/\partial r = -P(r,\alpha). \tag{A 6}$$

THEOREM 1. The quantities $\psi(r,\alpha)$ and $P(r,\alpha)$ (as well as $\phi(r,\alpha)$) are analytic functions of the complex variable α.

Proof. We use the following infinite series representations:

$$I_1(\alpha) = \sum_{n=0}^{\infty} (\tfrac{1}{2}\alpha)^{2n+1}/n!(n+1)!,$$

$$K_1(\alpha) = \alpha^{-1} + I_1(\alpha)\log\tfrac{1}{2}\alpha - \frac{1}{2}\sum_{n=0}^{\infty}(\tfrac{1}{2}\alpha)^{2n+1}(\tau(n+1)+\tau(n+2))/n!(n+1)!,$$

where $-\tau(1) = \gamma$ (the Euler–Mascheroni constant) and

$$\tau(n+1) = 1 + \tfrac{1}{2} + \ldots + 1/n - \gamma.$$

Thus, as is well known, $I_1(\alpha)$ and $\alpha^{-1}I_1(\alpha)$ are analytic while $K_1(\alpha)$ is singular at $\alpha = 0$. However, from (A 3) and the given series,

$$\psi(r,\alpha) = I_1(r\alpha)\, I_1(r_1\alpha)\log rr_1^{-1} + (r\alpha)^{-1} I_1(r_1\alpha) - (r_1\alpha)^{-1} I_1(r\alpha) + \frac{1}{2}\sum_{n=0}^{\infty}[r_1^{2n+1} I_1(r\alpha) - r^{2n+1} I_1(r_1\alpha)]\left(\frac{\alpha}{2}\right)^{2n+1}\frac{\tau(n+1)+\tau(n+2)}{n!(n+1)!}, \tag{A 7}$$

which, for one thing, involves neither a logarithmic singularity nor a pole at $\alpha = 0$. Contrary to expectations, then, $\psi(r,\alpha)$ is non-singular at $\alpha = 0$, but is in fact an analytic function of α. The analyticity of $P(r,\alpha)$ in α follows from (A 6) and, consequently, that of $\phi(r,\alpha)$ is implied by (A 1). *Q.E.D.*

Evidently, from (A 3), $\psi = \psi(r,\alpha)$ satisfies the Bessel equation

$$r\frac{\partial^2\psi}{\partial r^2}+\frac{\partial\psi}{\partial r}-\left(\alpha^2 r+\frac{1}{r}\right)\psi=0. \tag{A 8}$$

Note that

$$\psi(r,\alpha) = \psi(r,-\alpha) \quad \text{and} \quad P(r,\alpha) = P(r,-\alpha) \quad (\text{by (A 6)}); \tag{A 9}$$

$$\overline{\psi(r,\alpha)} = \psi(r,\bar{\alpha}), \quad \overline{P(r,\alpha)} = P(r,\bar{\alpha}), \tag{A 10}$$

the bar symbolizing a complex conjugate. Whence

$$\overline{\psi(r,i\,\mathrm{Im}\,\alpha)} = \psi(r,i\,\mathrm{Im}\,a), \quad \overline{P(r,i\,\mathrm{Im}\,\alpha)} = P(r,i\,\mathrm{Im}\,\alpha), \tag{A 11}$$

implying that $\psi(r,i\,\mathrm{Im}\,\alpha)$ and $P(r,i\,\mathrm{Im}\,\alpha)$ are real. We observe that (A 9) is also deducible from (A 7) and the $I_1(\alpha)$ series. We shall next investigate possible zeros of $P(r,\alpha)$ within the complex-α plane. But first, we need certain identities. Now, from (A 8), we have for any α_ν

$$(\alpha^2-\alpha_\nu^2)\,r\psi(r,\alpha)\,\psi(r,\alpha_\nu) = \frac{\partial}{\partial r}\left(r\psi(r,\alpha_\nu)\frac{\partial\psi}{\partial r}(r,\alpha)-r\psi(r,\alpha)\frac{\partial\psi}{\partial r}(r,\alpha_\nu)\right),$$

which can be integrated between the limits $r = r_0\ (>0)$, say, and $r = r_1\ (>r_0)$ by incorporating the fact that $\psi(r_1,\alpha) \equiv 0$ (see (A 3)) and (A 6). Thus,

$$(\alpha^2-\alpha_\nu^2)\int_{r_0}^{r_1} r\psi(r,\alpha)\,\psi(r,\alpha_\nu)\,dr = P(r_0,\alpha)\,\psi(r_0,\alpha_\nu)-P(r_0,\alpha_\nu)\,\psi(r_0,\alpha). \tag{A 12}$$

So, at $\alpha = \overline{\alpha_\nu}$,

$$2(\mathrm{Re}\,\alpha_\nu)\,(\mathrm{Im}\,\alpha_\nu)\int_{r_0}^{r_1} r|\psi(r,\alpha_\nu)|^2\,dr = \mathrm{Im}\,[P(r_0,\alpha_\nu)\,\overline{\psi(r_0,\alpha_\nu)}], \tag{A 13}$$

via (A 10). Also, on replacing α_ν by $\overline{\alpha_\nu}$ in the expression (A 12) and then differentiating the result with respect to α, we obtain at $\alpha = \alpha_\nu$

$$\begin{aligned}2\alpha_\nu\int_{r_0}^{r_1} r|\psi(r,\alpha_\nu)|^2\,dr+4i(\mathrm{Re}\,\alpha_\nu)\,(\mathrm{Im}\,\alpha_\nu)\int_{r_0}^{r_1} r\psi_\alpha(r,\alpha_\nu)\,\overline{\psi(r,\alpha_\nu)}\,dr \\ = P_\alpha(r_0,\alpha_\nu)\,\overline{\psi(r_0,\alpha_\nu)}-\overline{P(r_0,\alpha_\nu)}\,\psi_\alpha(r_0,\alpha_\nu),\end{aligned} \tag{A 14}$$

where the suffix α denotes an α derivative. Note that, for any α_ν, $\psi(r,\alpha_\nu)$ is not identically zero throughout $r_0 \leqslant r \leqslant r_1$.

Suppose that $P(r_0,\alpha_\nu) = 0$. Whereupon, (A 13) implies that either $\mathrm{Re}\,\alpha_\nu = 0$, or $\mathrm{Im}\,\alpha_\nu = 0$, or $\alpha_\nu = 0$. Then, from (A 14),

$$P_\alpha(r_0,\alpha_\nu)\,\overline{\psi(r_0,\alpha_\nu)} = 2\alpha_\nu\int_{r_0}^{r_1} r|\psi(r,\alpha_\nu)|^2\,dr, \tag{A 15}$$

which is non-vanishing unless $\alpha_\nu = 0$; i.e. $P_\alpha(r_0,\alpha_\nu) \neq 0$ and $\psi(r_0,\alpha_\nu) \neq 0$ unless $\alpha_\nu = 0$. Also, from (A 7) and the $I_1(\alpha)$ series,

$$\psi(r,0) = (r_1^2-r^2)/2r_1 r \neq 0 \quad \text{if} \quad r \neq r_1, \tag{A 16}$$

so that, if $\alpha_\nu = 0$ in (A 15), then the only possibility is $P_\alpha(r_0,0) = 0$. Thus we can state theorem 2.

THEOREM 2. Any (existing) α-zero of $P(r_0, \alpha)$ is either purely real and simple, or purely imaginary and simple, or appears as a repeated zero at the origin $\alpha = 0$. Furthermore, any such α-zero of $P(r_0, \alpha)$ is never an α-zero of $\psi(r_0, \alpha)$.

Note. Letting $\alpha_\nu = 0$ and assuming $P(r_0, 0) = 0$ the second α derivative of (A 12) yields at $\alpha = 0$

$$P_{\alpha\alpha}(r_0, 0)\,\psi(r_0, 0) = 2\int_{r_0}^{r_1} r[\psi(r, 0)]^2\,dr \neq 0$$

owing to (A 16). Therefore, if the zero at $\alpha = 0$ arises, it does so as a double zero.

We now go on to establish theorem 3.

THEOREM 3. The function $P(r_0, \alpha)$ has

(i) exactly two real, symmetric (simple) zeros $\alpha = \pm\alpha_0$ if $\lambda > \chi(0)$,

(ii) the double zero at $\alpha = 0$ if $\lambda = \chi(0)$,

(iii) no real zero if $\lambda < \chi(0)$, where

$$\chi(\alpha) = \phi(r_0, \alpha)/\psi(r_0, \alpha). \tag{A 17}$$

Proof. For a real positive argument α, the Bessel functions $I_1(\alpha)$ and $K_1(\alpha)$ are known to be both positive; furthermore, they are monotonically increasing and decreasing, respectively, with increasing α (see e.g. Gray & Mathews 1922, chap. 7), i.e.

$$I_1(r_1\alpha) > I_1(r\alpha) > 0, \quad K_1(r\alpha) > K_1(r_1\alpha) > 0,$$

whenever $r_0 \leqslant r < r_1$ and α is real and > 0. In this case, and by virtue of (A 3),

$$\psi(r, \alpha) > 0, \tag{A 18}$$

which is also valid at $\alpha = 0$ owing to (A 16). Now, applying the form (A 1) to (A 12), we can express in terms of $\chi(\alpha)$

$$(\alpha^2 - \alpha_\nu^2)\frac{\displaystyle\int_{r_0}^{r_1} r\psi(r, \alpha)\,\psi(r, \alpha_\nu)\,dr}{\psi(r_0, \alpha)\,\psi(r_0, \alpha_\nu)} = \chi(\alpha) - \chi(\alpha_\nu).$$

On the left side, $\alpha^2 - \alpha_\nu^2$ is multiplied by a factor which is clearly positive whenever α and α_ν are real and $\geqslant 0$. Whereupon, the analytic function (of a real α)

$$\chi(\alpha) > \chi(\alpha_\nu) \quad \text{for all} \quad \alpha > \alpha_\nu \geqslant 0.$$

Therefore, in the α, κ plane, the curve $\kappa = \chi(\alpha)$ is continuous, smooth and is monotonically ascending with increasing positive α from the point $(0, \chi(0))$. Hence, if this curve cuts the horizontal line $\kappa = \lambda$, then it does so exactly once within $\alpha > 0$. Obviously, for such an intersection to occur, it is first necessary that $\lambda > \chi(0)$. This condition alone is also sufficient, because $\kappa = \chi(\alpha)$ is not bounded above by a horizontal asymptote as can be seen from the asymptotic approximation (cf. §4) $\chi(\alpha) \sim r_0\alpha$ as $\alpha \to +\infty$. If $\lambda < \chi(0)$, there is no intersection within $\alpha > 0$. But the abscissa of any existing intersection in $\alpha > 0$ is, via (A 1) and (A 17), a real positive α-zero of $P(r_0, \alpha)$, and vice versa. Furthermore, in view of (A 9), the α-zeros of $P(r_0, \alpha)$ must arise in symmetric pairs about $\alpha = 0$. Consequently, statements (i) and (iii) of the theorem follow. The (double) zero at $\alpha = 0$ appears if the line $\kappa = \lambda$ meets the curve $\kappa = \chi(\alpha)$ along the κ axis, i.e. if $\lambda = \chi(0)$. This establishes statement (ii), and completes our proof.

In terms of the fundamental Bessel functions $J_1(\kappa)$ and $Y_1(\kappa)$

$$I_1(i\kappa) = iJ_1(\kappa), \quad K_1(i\kappa) = -\tfrac{1}{2}\pi(J_1(\kappa) - iY_1(\kappa)).$$

From (A 3), then,

$$\psi(r_0, i\kappa) = \tfrac{1}{2}\pi[J_1(r_0\kappa)Y_1(r_1\kappa) - Y_1(r_0\kappa)J_1(r_1\kappa)], \tag{A 19}$$

which is real whenever κ is real, consistent with (A 11). Now, the κ-zeros of the square-bracketed expression in (A 19) have been analysed (Gray & Mathews *op. cit.*). There is an infinity of these zeros at $\kappa = \pm|\kappa_\nu|$ ($\nu = 1, \ldots, \infty$; $\kappa_\nu \neq 0$), all of which are real and simple, i.e. $\psi(r_0, \alpha)$ has only simple imaginary zeros at $\alpha = \pm i|\kappa_\nu|$ ($\nu = 1, \ldots, \infty$).

THEOREM 4. Suppose that the entire set of zeros of $\psi(r_0, \alpha)$ at $\alpha = \pm i|\kappa_\nu|$ ($\nu = 1, \ldots, \infty$) are arranged as a monotonic sequence:

$$|\kappa_1| < |\kappa_2| < \ldots < |\kappa_n| < |\kappa_{n+1}| < \ldots.$$

Then whatever the value of λ, $P(r_0, \alpha)$ has an infinity of symmetric zeros at

$$\alpha = \pm i|\alpha_\nu| \quad (\nu = 1, \ldots, \infty), \tag{A 20}$$

satisfying

$$|\kappa_1| < |\alpha_1| < |\kappa_2| < |\alpha_2| < \ldots < |\kappa_n| < |\alpha_n| < |\kappa_{n+1}| < \ldots. \tag{A 21}$$

(i) If $\lambda > \chi(0)$ (with $\chi(0)$ determined from (A 17)), the entire set of imaginary α-zeros of $P(r_0, \alpha)$ is fully definable by (A 20) and (A 21).

(ii) If $\lambda < \chi(0)$, the entire set of imaginary α-zeros of $P(r_0, \alpha)$ is definable by (A 20) and (A 21) together with two additional zeros at $\alpha = \pm i|\alpha_0|$ satisfying

$$-|\kappa_1| < -|\alpha_0| < 0 < |\alpha_0| < |\kappa_1|. \tag{A 22}$$

Remarks. Actually, $\chi(0) = 2r_0^2/(r_1^2 - r_0^2)$ (see (3.13)). This theorem improves upon the last sentence of theorem 2, and provides an approximate location of those (simple) zeros of $P(r_0, \alpha)$ along the $\operatorname{Im}\alpha$ axis relative to the known zeros of $\psi(r_0, \alpha)$, viz. between any two zeros of the latter lies one zero of the former except for the two zeros $\pm i|\kappa_1|$ of $\psi(r_0, \alpha)$ nearest the origin. Between these two, $P(r_0, \alpha)$ is non-vanishing for imaginary α whenever $\lambda > \chi(0)$, but vanishes twice if $\lambda \leqslant \chi(0)$, with equality corresponding to two coincident zeros forming a double zero at $\alpha = 0$ (see statement (ii) of theorem 3). Apparently, then, as λ approaches $\chi(0)$ from below, the α-zeros $\pm i|\alpha_0|$ converge vertically towards coincidence at $\alpha = 0$. They then diverge laterally, transformed into real zeros $\pm\alpha_0$, along the $\operatorname{Re}\alpha$ axis once λ exceeds the level $\chi(0)$ (cf. theorem 3). Though all other imaginary α-zeros of $P(r_0, \alpha)$ do move with this λ-variation, nevertheless they never stray off the $\operatorname{Im}\alpha$ axis, and they remain governed by (A 21).

Proof. The symmetry distribution of all α-zeros is a consequence of (A 9). Now, substitution into (A 14) of α_ν by the α-zero $i|\kappa_\nu|$ of $\psi(r_0, \alpha)$ yields, for $\nu = 1, \ldots, \infty$,

$$P(r_0, i|\kappa_\nu|)\left[\frac{\partial\psi}{\partial\kappa}(r_0, i\kappa)\right]_{\kappa=|\kappa_\nu|} = 2|\kappa_\nu| \int_{r_0}^{r_1} r|\psi(r, i|\kappa_\nu|)|^2\, dr > 0, \tag{A 23}$$

both factors to the left of the equals sign being real (cf. (A 11)). At two consecutive positive κ-zeros of $\psi(r_0, i\kappa)$, the derivative $\partial\psi(r_0, i\kappa)/\partial\kappa$ assumes opposite signs, and so does $P(r_0, i\kappa)$ by virtue of (A 23). Therefore, between any two amongst the infinity of positive consecutive κ-zeros of $\psi(r_0, i\kappa)$, $P(r_0, i\kappa)$ vanishes at least once for real κ. In particular, then, $P(r_0, \alpha)$ does possess an infinity of imaginary α-zeros at, say, $\alpha = \pm i|\alpha_\nu|$ $(\nu = 1, \ldots, \infty)$. Whereupon, from (A 15),

$$\psi(r_0, i|\alpha_\nu|)\left[\frac{\partial P}{\partial \kappa}(r_0, i\kappa)\right]_{\kappa=|\alpha_\nu|} = -2|\alpha_\nu|\int_{r_0}^{r_1} r|\psi(r, i|\alpha_\nu|)|^2\,dr < 0. \tag{A 24}$$

Whence, by an argument similar to that above, $\psi(r_0, i\kappa)$ takes opposite signs at any two consecutive positions $\kappa = |\alpha_\nu|$ and $|\alpha_{\nu+1}|$, so that it does vanish between these positions. Suppose that $|\alpha_\nu| < |\alpha_{\nu+1}|$ $(\nu = 1, \ldots, \infty)$. If we assume that $\psi(r_0, i\kappa)$ vanishes more than once in $|\alpha_\nu| < \kappa < |\alpha_{\nu+1}|$, we arrive at a contradiction of the proposition that $|\alpha_\nu|$ and $|\alpha_{\nu+1}|$ are consecutive. Hence, $\psi(r_0, i\kappa)$ vanishes exactly once in $|\alpha_\nu| < \kappa < |\alpha_{\nu+1}|$. Likewise, $P(r_0, i\kappa)$ vanishes exactly once in $|\kappa_\nu| < \kappa < |\kappa_{\nu+1}|$. Therefore the statement including (A 20) and (A 21) is certainly true. In this case, $\alpha = \pm i|\alpha_\nu|$ $(\nu = 1, \ldots, \infty)$ constitute the only imaginary zeros of $P(r_0, i\alpha)$ beyond the segment $-|\kappa_1| \leqslant \mathrm{Im}\,\alpha \leqslant |\kappa_1|$ of the $\mathrm{Im}\,\alpha$ axis.

From (A 1) and (A 17), we have

$$P(r_0, i\kappa) = \psi(r_0, i\kappa)\,[\chi(i\kappa) - \lambda]. \tag{A 25}$$

From (A 16),

$$\psi(r_0, 0) = (r_1^2 - r_0^2)/2r_1 r_0 > 0. \tag{A 26}$$

Case $\lambda > \chi(0)$. Here, $P(r_0, 0) < 0$, so that $\partial P(r_0, i\kappa)/\partial\kappa > 0$ at $\kappa = \alpha^*$, the smallest positive κ-zero of $P(r_0, i\kappa)$. Hence via (A 24), $\psi(r_0, i\alpha^*) < 0$, implying, when coupled with the inequality of (A 26), that $\psi(r_0, i\kappa)$ has just one positive κ-zero, precisely, its smallest positive κ-zero $|\kappa_1|$, inside $0 < \kappa < \alpha^*$. By virtue of (A 21), $\alpha^* = |\alpha_1|$. Statement (i) of the theorem is thus verified.

Case $\lambda < \chi(0)$. In view of (A 26), $\partial\psi(r_0, i\kappa)/\partial\kappa < 0$ at $\kappa = \min_\nu |\kappa_\nu| \equiv |\kappa_1|$. So, (A 23) reveals that $P(r_0, i|\kappa_1|) < 0$. But $P(r_0, 0) > 0$ by (A 25) and (A 26). Hence, inside $0 < \kappa < |\kappa_1|$, $P(r_0, i\kappa)$ has exactly one zero which obviously is not $|\alpha_1|$, but must be a zero $\kappa = |\alpha_0|$ additional to the set $|\alpha_\nu|$ $(\nu = 1, \ldots, \infty)$. This proves the statement (ii). *Q.E.D.*

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